Theme:The fundamental solution of the fraktional diffusion equation

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Abstract: This paper provides information on the fundamental solution of the fractional diffusion equation.

Keywords: diffusion eqution, Koshi issue,fundamental solution, Fourier transform, Mittag-Lefler function.

$$\begin{cases} {}^{K}D_{0+t}^{\alpha} y(x) + \lambda y(x) = f(x) \quad x \epsilon R, t > 0 \\ y(0) = y_o \end{cases}$$ (1) → Koshi (1) issue

y(x)=$y_0 E_{\alpha}(-\lambda x^{\alpha})+\int_0^x y^{\alpha-1} E_{\alpha,\alpha}(-\lambda y^{\alpha}) f(x-y) dy$ (2) → The solution to the Koshi(1) problem.

We derive the fundamental solution of the fractional diffusion equation by the solution of the Koshi(1) problem.

The following question is posed:

$$\begin{cases} {}^{K}D_{0+t}^{\alpha} U(x,t) - U_{xx} = f(x,t) \quad x \epsilon R, t > 0 \\ U(x,0) = \varphi(x) \end{cases} \quad (3)$$

(3)We use Fourier integral substitution to find the solution of the equation.

Fourier substitution of the function f in R

F[f]=F[f(x)](ʒ) $= \hat{f}(ʒ) := \int_{-\infty}^{+\infty} e^{ixʒ} f(x) dx, ʒ \epsilon$R appears.

We use the integral Fourier substitution for a fractional order product meaning Caputo:

$$F\left[{}^{K}D_{0+t}^{\alpha} U\right] = {}^{K}D_{0+t}^{\alpha} \hat{U}$$

We also express U_{xx} by the Fourier integral substitution:

$F[U_{xx}]=\int_{-\infty}^{+\infty} e^{ixʒ} U_{xx} dx=\int_{\infty}^{-\infty} e^{-ixʒ} d(U_x)=$

$=e^{-ixʒ} U_x|_{-\infty}^{\infty} + ʒi \int_{+\infty}^{-\infty} e^{-ixʒ} U_x dx = 0 + ʒi \int_{-\infty}^{\infty} e^{-ixʒ} dU = ʒi(e^{ixʒ} U|_{-\infty}^{\infty} +$

$+ ʒi \int_{\infty}^{-\infty} e^{ixʒ} U d_x = ʒi(0 + iʒ\hat{U}) = - ʒ^2 \hat{U}$

(When it comes to solutions $\int udv = uv - \int vdu$ from the fractional integration formula,

$e^{-ix} = cosx - isinx$, $\lim_{|x|\to\infty}(U_x; U) = 0$ we used the equations.)

$$F[f(x,t)]\,(ʒ) = \hat{f}(ʒ, t)$$

If we put the resulting expressions in equation(3),this equation takes the following form:

$$\begin{cases} {}^{K}D_{0+t}^{\alpha}\widehat{U} + ʒ^2\widehat{U} = \hat{f}(ʒ,t) \quad ʒ\epsilon R, t>0 \\ \widehat{U}(ʒ,0) = \hat{\varphi}(ʒ) \end{cases} \quad (4)$$

The view of the equation is consistent with Koshi(1) problem:

$$\begin{cases} {}^{K}D_{0+t}^{\alpha}y(x) + \lambda y(x) = f(x) \quad x\epsilon R, t>0 \\ y(0) = y_o \end{cases} \quad (1)$$

To find the solution of equation(4),we use the solution of Koshi problem(1)

$y(x)=y_0E_\alpha(-\lambda x^\alpha)+\int_0^x y^{\alpha-1}E_{\alpha,\alpha}(-\lambda y^\alpha)f(x-y)dy$ expression(1) is solution of equation.

Accordingly,we give the solution of equation(4):

$$\widehat{U}(ʒ;t) = E_\alpha(-ʒ^2t^\alpha) + \int_0^t y^{\alpha-1}E_{\alpha,\alpha}(-ʒ^2y^\alpha)\hat{f}(ʒ;t-y)dy \quad (5)$$

According to this equation $\begin{cases} {}^{K}D_{0+t}^{\alpha}U(x,t) - U_{xx} = f(x,t) \quad x\epsilon R, t>0 \\ U(x,0) = \varphi(x) \end{cases} \quad (3)$

we use the inverse Fourier substitution to achieve the solution of equation(3).

The inverse Fourier substitution of the function f(x) is called the following integral:

$$F^{-1}[f] = F^{-1}[f(x)](ʒ) = \frac{1}{2\pi}\int_R e^{ixʒ}f(ʒ)dʒ$$

Ergo : $U=F^{-1}[\widehat{U}] = \frac{1}{2\pi}\int_R e^{ixʒ}\hat{u}(ʒ;t)dʒ \quad (6)$

Substituting (5) for $\widehat{U}$ in equation(6),we obtain a fundamental solution of the fractional diffusion equation of the form:

$$U(x,t)=\frac{1}{2\pi}\int_R e^{ixʒ}\left[E_\alpha(-ʒ^2t^\alpha) + \int_0^t y^{\alpha-1}E_{\alpha,\alpha}(-ʒ^2y^\alpha)\hat{f}(ʒ;t-y)dy\right]dʒ \quad (7)$$

Eqution(7) is a solution of eqution(3).

We consider that this solution is appropriate for the following issue:

$$\begin{cases} {}^{K}D_{0+t}^{\alpha}U(x,t) - U_{xx} = f \quad x\epsilon R, t > 0 \\ U(x,0) = \delta(x) \end{cases} \quad (8)$$

Here δ(x) is Dirac's delta function:

$$\delta(x - x_o) = \begin{cases} +\infty, & x = x_o \\ 0, & x \neq x_o \end{cases}$$

$$\int_R \delta(x - x_o)dx = 1 \quad , \quad \forall y \in D(R) = C_0^{\infty}(\mathrm{R})$$

$$\int_R \delta(x - x_o)\varphi(x)dx = \varphi(x_0) \ (9) \quad , \quad \delta(x) = \delta(-x) \ \text{(dual function)}$$

Find U(x,0) by equation(7):

$$\mathrm{U(x,0)}=\frac{1}{2\pi}\int_R e^{ix\mathfrak{z}}\left[E_{\alpha}(-\mathfrak{z}^2t^{\alpha}) + \int_0^0 y^{\alpha-1}E_{\alpha,\alpha}(-\mathfrak{z}^2y^{\alpha})\hat{f}(\mathfrak{z};0-y)dy\right]d\mathfrak{z} =$$

$$\frac{1}{2\pi}\int_R e^{ix\mathfrak{z}}d\mathfrak{z} = \delta(x)$$

(This is the result $\int_0^0 y^{\alpha-1}E_{\alpha,\alpha}(-\mathfrak{z}^2y^{\alpha})\hat{f}(\mathfrak{z};0-y)dy = 0$,

$E_{\alpha}(-\mathfrak{z}^2t^{\alpha}) = \sum_{n=0}^{\infty}\frac{(-\mathfrak{z}^2t^{\alpha})^n}{\Gamma(\alpha n+1)}$ originated according to the expressions)

According to formula (9) it follows that the solution expressed by eqution(7) is appropriated for the problem of the form(8)

$$\mathrm{U(x,0)}=\int_R \varphi(y)[\frac{1}{2\pi}\int_R e^{i\mathfrak{z}(x-y)}d\mathfrak{z}]dy = \int_R \varphi(y)\ \delta(x-y)dy = \varphi(x)$$

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